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Partitioning the Σ-products

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Abstract

This paper addresses the topological partition relations of the form $2^{\omega_1} \to (\omega_1 + 1)^1_2$ and $\Sigma_{\aleph_2}\{0,1\} \to (\omega_1)^1_n$, and in the latter case it completes the picture. © 2000 Elsevier Science B.V. All rights reserved.

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Introduction

Topological partition calculus is a natural extension of partition calculus, see, for example, Weiss [8].

Notation. For topological spaces X and Y and integer $n \geqslant 1$, $X \to (Y)^1_n$ stands for the following statement: "For any partition of X into n pieces there is one piece of the partition that contains a topological copy of the space Y"; this copy of the space Y is called a *homogeneous* copy of Y. The negation of this statement is denoted by $X \nrightarrow (Y)^1_n$. Similarly we consider $X \to (Y, Z)^1$ and $X \nrightarrow (Y, Z)^1$.

This article addresses the interesting behavior of the case when $X = \Sigma_{\aleph_2}\{0,1\}$, and $Y = \omega_1$, the first uncountable ordinal space. The Sigma product $\Sigma_{\aleph_2}\{0,1\}$ is the subspace of $\prod_{\omega_2}\{0,1\}$, hereafter denoted by 2^{ω_2}, whose elements have countable supports. For properties of the Σ-products we refer the reader to Engelking [2].

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It is easy to see that ω_1 embeds into $\Sigma_{\aleph_2}\{0,1\}$ but ω_1+1 does not, and that $\omega_1 \to (\omega_1,\alpha)^1$ for any countable ordinal α. These imply that

$$\Sigma_{\aleph_2}\{0,1\} \to (\omega_1,\alpha)^1 \quad \text{and} \quad \Sigma_{\aleph_2}\{0,1\} \nrightarrow (\omega_1+1)^1_1.$$

The consistency of the following positive relation is established by Shelah [6].

$$\Sigma_{\aleph_2}\{0,1\} \to (\omega_1)^1_2.$$

We establish the consistency of the corresponding negative relation using a forcing argument (see Corollary 4). In Shelah's consistency proof of the positive relation the axiom SPFA is used (see Section 4), which is a large cardinal assumption. We explore the consistency strength this positive relation by demonstrating that this assumption implies Chang's Conjecture (Proposition 2). We further establish the following negative relation in ZFC (Proposition 3),

$$\Sigma_{\aleph_2}\{0,1\} \nrightarrow (\omega_1)^1_3. \tag{$\dagger$}$$

In another direction Velickovic translated the above result of Shelah to prove the consistency of $2^{\omega_1} \to (\omega_1)^1_2$. Juhasz asked (see a survey article of Weiss [8]) if it is consistent that $2^{\omega_1} \to (\omega_1+1)^1_2$. To answer this question partially we introduce a lemma due to Todorcevic (Lemma 2) to extend the result of Velickovic to (Proposition 1):

$$\text{(SPFA)} \quad 2^{\omega_1} \to \big(\omega_1+1,\, B(\aleph_2)\big)^1, \tag{$\ddagger$}$$

where $B(\aleph_2)$ is the Baire space of weight $\aleph_2$, which contains among others, spaces ω_1 and the Cantor set 2^ω. Of course, the Baire space does not contain the space ω_1+1, hence the question of Juhasz remains open.

The elementary submodels are heavily used in this paper, hence we start with an introduction to elementary submodels (Section 1), then we will introduce the *Reflection Principle* and prove a crucial lemma for the later constructions. Section 3 translates the structures of Section 2 to topological objects, to be used in our SPFA argument. Section 4 contains a modification of Shelah's SPFA argument to prove (‡). Section 5 introduces Chang's Conjecture and proves that the positive relation $\Sigma_{\aleph_2}\{0,1\} \to (\omega_1)^1_2$ is indeed a large cardinal assumption. Section 6 applies the technique of Section 5 to produce the negative result (†). Finally, Section 7 includes our forcing argument to establish the consistency of the negative relation $\Sigma_{\aleph_2}\{0,1\} \nrightarrow (\omega_1)^1_2$ among other negative results (see Corollary 4).

1. Elementary submodels

In this section we introduce and explore the notion of *elementary submodels* of some structure of the form $\langle H_\theta, \in\rangle$, where $\boldsymbol{H}_\theta$ is the set of all the sets whose transitive closure has cardinality less than θ.

Definition 1. A submodel M of a model N is elementary, denoted by $M \prec N$, if for any formula ϕ with parameters from M, we have that $M \models \phi$ iff $N \models \phi$.

The following result is used to establish the elementarity of submodels of $\boldsymbol{H}_\theta$.

Tarski–Vaught test. *For a submodel M of N, we have $M \prec N$ iff for each formula $\phi(y, x_1, \ldots, x_n)$ with parameters $x_1, \ldots, x_n$ from M*

$$N \models \exists y \phi(y, x_1, \ldots, x_n) \Rightarrow (\exists y \in M) N \models \phi(y, x_1, \ldots, x_n).$$

Notations.

(1) For a cardinal θ, $\boldsymbol{H}_\theta$ denotes collection of all sets whose transitive closure has size less than θ. It is known that for a regular θ, $(\boldsymbol{H}_\theta, \in)$ is a transitive model of axioms of ZFC other than the *power set axiom* (see Kunen [5]).

(2) For any formula ϕ, $\phi^{\boldsymbol{H}_\theta}$ denotes the relativization of the formula ϕ to $\boldsymbol{H}_\theta$, and it means $\boldsymbol{H}_\theta \models \phi$. It is known that $\phi^{\boldsymbol{H}_\theta}$ is absolute for transitive models that contain $\boldsymbol{H}_\theta$, in particular, if $\lambda > \theta$, then $(\boldsymbol{H}_\lambda \models \phi^{\boldsymbol{H}_\theta})$ iff $\phi^{\boldsymbol{H}_\theta}$ (see Kunen [5]).

(3) We shall frequently and implicitly add a well ordering $\leqslant_w$ of $\boldsymbol{H}_\theta$ as a predicate and we consider $(\boldsymbol{H}_\theta, \in, \leqslant_w)$ instead of $(\boldsymbol{H}_\theta, \in)$, so that for a given set A, we can take $Sk_\theta(A)$, the Skolem closure of A in $(\boldsymbol{H}_\theta, \in, \leqslant_w)$.

(4) For elementary submodels $M \prec N$, we say that N *end-extends* M if

$$M \cap \omega_1 = N \cap \omega_1.$$

Our first elementary submodel fact can be derived from Tarski–Vaught test:

Lemma 1. *Suppose $M \prec H_\lambda$, $\theta < \lambda$ and $\theta \in M$. Then, $H_\theta \in M$ and $M \cap H_\theta \prec H_\theta$.*

We shall also need some standard facts about the notions of stationary sets and closed-unbounded sets (clubs) in the structures of the form $[A]^{\aleph_0}$. The following facts will be occasionally used (see Jech [3]):

(1) The collection of all countable elementary submodels of a $\boldsymbol{H}_\theta$ forms a club in $[\boldsymbol{H}_\theta]^{\aleph_0}$.

(2) (a) If C is club in $[A]^{\aleph_0}$ then $C^* = \{X \in [B]^{\aleph_0}\colon X \cap A \in C\}$, contains a club in $[B]^{\aleph_0}$, and if C is club in $[B]^{\aleph_0}$ then $C|_A = \{x \cap A\colon x \in C\}$, contains a club of $[A]^{\aleph_0}$.

(b) If $A \subseteq B$ and $S \subseteq [A]^{\aleph_0}$ is stationary, then S^* is stationary in $[B]^{\aleph_0}$. Conversely, if S is stationary in $[B]^{\aleph_0}$ then $S|_A$ is stationary in $[A]^{\aleph_0}$.

(3) For any set A,

(a) if $F : [A]^{<\omega} \to [A]^\omega$, then the set

$$\mathcal{C}_F = \left\{x \in [A]^\omega\colon (\forall e \in [x]^{<\omega})\ F(e) \subseteq x\right\},$$

is a club.

(b) (Kueker's lemma) For every club $\mathcal{C}$ in $[A]^\omega$ there is a function

$$F : [A]^{<\omega} \to [A]^\omega$$

such that $\mathcal{C}_F \subseteq \mathcal{C}$.

2. Reflection Principle

The Reflection Principle (RP) is the following statement (see Bekkali [1]):

Reflection Principle. *For every cardinal κ, any set A of size κ, every stationary $S \subseteq [A]^{\aleph_0}$, every cardinal $\lambda > \kappa$ and any countable $M_0 \prec \boldsymbol{H}_\lambda$, there is a continuous $\in$-chain $\{M_\xi\colon \xi < \omega_1\}$ of countable elementary submodels of $\boldsymbol{H}_\lambda$ which starts from M_0, and an stationary subset $E \subseteq \omega_1$ such that $(\forall \xi \in E)\ M_\xi \cap A \in S$.*

Given a countable elementary submodel M, and some ordinal $\alpha \notin M$, we often extend M to an elementary submodel that extends M and contains α. In doing so many new objects that may not be desirable are also added. We use RP to provide an abundance of some carefully chosen ordinals so that extensions of M using these ordinals do not include undesirable ordinals.

In the following discussions we assume $\theta = (2^{\aleph_2})^+$ and $\lambda = (2^{2^{\aleph_2}})^+$. For any countable elementary submodel M of $\boldsymbol{H}_\lambda$ define

$$D_M = \left\{\alpha < \omega_2\colon (\forall f \in M \cap {}^{\omega_2}\omega_2)(f \text{ is regressive } \Rightarrow f(\alpha) \in M)\right\}.$$

Lemma 2 (RP). *The collection*

$$\mathcal{T} = \left\{M \prec \boldsymbol{H}_\theta\colon D_M \textit{ is unbounded in } \omega_2\right\}$$

contains a club subset of $[\boldsymbol{H}_\theta]^{\aleph_0}$.

Proof. Suppose $\mathcal{S} = [\boldsymbol{H}_\theta]^{\aleph_0} \setminus \mathcal{T}$ is stationary. By RP find a continuous $\in$-chain $\{M_\xi\colon \xi < \omega_1\}$ of countable elementary submodels of $\boldsymbol{H}_\lambda$ with $\theta \in M_0$, and a stationary $E \subseteq \omega_1$ such that $(\forall \xi \in E)\ M_\xi \cap \boldsymbol{H}_\theta \in \mathcal{S}$. Let

$$\delta = \bigcup_{\xi < \omega_1} (M_\xi \cap \omega_2).$$

Since the chain of submodels is an $\in$-chain, we have for each ξ, that $M_\xi \cap \omega_1 < M_{\xi+1} \cap \omega_1$, so that $\omega_1 \subseteq \bigcup_{\xi<\omega_1} M_\xi$. Hence, any $\alpha < \delta$ will fall in $\overline{M} = \bigcup_{\xi\in\omega_1} M_\xi$ as $\overline{M}$ is an elementary submodel and $\omega_1 \subseteq \overline{M}$. Therefore, δ is an ordinal.

Take a countable $N \prec \boldsymbol{H}_\lambda$ such that $\{M_\xi\colon \xi < \omega_1\} \in N$ and $\delta, \theta \in N$ and $\eta = N \cap \omega_1 \in E$. This is possible because there are club many $N \prec \boldsymbol{H}_\theta$ which contain $\{M_\xi\colon \xi < \omega_1\}$, by fact (1), and from fact (2),

$$E^* = \left\{X \in [\boldsymbol{H}_\lambda]^{\aleph_0} =\colon X \cap \omega_1 \in E\right\}$$

is stationary in $[\boldsymbol{H}_\lambda]^{\aleph_0}$. $\square$

Observation. For each $\xi \in E$ we have $M_\xi \cap \boldsymbol{H}_\theta \in \mathcal{S}$, hence by the definition of $\mathcal{S}$, $D_{M_\xi \cap \boldsymbol{H}_\theta}$ is bounded in ω_2. Let $\delta_\xi = \sup D_{M_\xi \cap \boldsymbol{H}_\theta}$, hence $\delta_\xi < \omega_2$. Now, $M_\xi \in M_{\xi+1}$ and $\boldsymbol{H}_\theta \in M_{\xi+1}$, so that $D_{M_\xi \cap \boldsymbol{H}_\theta} \in M_{\xi+1}$. It follows that $\delta_\xi \in M_{\xi+1} \cap \omega_2$, hence $\delta_\xi < \delta$.

Claim 1. $N \cap \bigcup_{\xi<\omega_1} M_\xi = M_\eta$.

Proof. $\eta = \omega_1^N$ is a limit ordinal and $\bigcup_{\xi<\eta} M_\xi = M_\eta$. Clearly, if $\xi < \eta$ then $M_\eta \subseteq N$, so that $M_\eta \subseteq N \cap \bigcup_{\xi<\eta} M_\xi$. On the other hand, let $x \in N \cap \bigcup_{\xi<\eta} M_\xi$. There is $\xi < \omega_1$ such that $x \in M_\xi$:

$$\begin{aligned} \boldsymbol{H}_\lambda &\models \text{“}(\exists \xi < \omega_1)\ x \in M_\xi\text{”} \Rightarrow N \models \text{“}(\exists \xi < \omega_1)\ x \in M_\xi\text{”} \\ &\Rightarrow (\exists \xi < \omega_1 \cap N)\ N \models \text{“}x \in M_\xi\text{”} \Rightarrow (\exists \xi < \eta)\ \boldsymbol{H}_\lambda \models \text{“}x \in M_\xi\text{”} \\ &\Rightarrow (\exists \xi < \eta)\ x \in M_\xi \subseteq M_\eta. \quad \Box \end{aligned}$$

Claim 2. $\delta = \min(N \cap \omega_2 \setminus M_\eta)$.

Proof. We need to show that if $\alpha < \delta$ and $\alpha \in N \cap \omega_2$ then $\alpha \in M_\eta$. If $\alpha \in N \cap \delta$, then $N \models \text{“}(\exists \xi < \omega_1)\ \alpha \in M_\xi \cap \omega_2\text{”}$. Hence, $(\exists \xi < \omega_1 \cap N)\ \alpha \in M_\xi \cap \omega_2$, i.e., $\alpha \in M_\xi \cap \omega_2$ for some $\xi < \eta$. $\Box$

Claim 3. $\delta \in D_{M_\eta \cap \boldsymbol{H}_\theta}$.

Proof. If $f \in M_\eta \cap \boldsymbol{H}_\theta \cap {}^{\omega_2}\omega_2$ is regressive then $f \in N$. As $\delta \in N$, we have $f(\delta) \in N \cap \delta$, hence by Claim 2 we have $f(\delta) \in M_\eta \cap \omega_2$. It follows that $\delta \in D_{M_\eta \cap \boldsymbol{H}_\theta}$. $\Box$

Claim 3 implies that $\delta < \delta_\eta$, contradicting the fact that $\delta_\eta < \delta$.

Corollary 1. *For any countable $M \prec \boldsymbol{H}_\lambda$ with $\theta \in M$, D_M is unbounded in ω_2.*

Proof. Let $\mathcal{C} \subseteq \mathcal{T}$ be a club. From Claim 3 we have

$$\boldsymbol{H}_\lambda \models \text{“}\big(\exists F : [\boldsymbol{H}_\theta]^{<\omega} \to [\boldsymbol{H}_\theta]^{\omega}\big)\ \mathcal{C}_F \subseteq \mathcal{C}\text{”}.$$

By elementarity of M and the fact that $\theta \in M$ we have

$$(\exists F \in M)\ \big(F : [\boldsymbol{H}_\theta]^{<\omega} \to [\boldsymbol{H}_\theta]^{\omega} \text{ and } \boldsymbol{H}_\lambda \models \text{“}\mathcal{C}_F \subseteq \mathcal{C}\text{”}\big).$$

$M \cap \boldsymbol{H}_\theta \in \mathcal{C}_F$, because it is closed under F; hence it is in $\mathcal{C}$, hence $D_{M \cap \boldsymbol{H}_\theta}$ is unbounded in ω_2.

The following argument shows that D_M must also be unbounded in ω_2.

We observe that any $f \in M \cap^{\omega_2} \omega_2$ is already in $\boldsymbol{H}_\theta$ because $|tc(f)| < (2^{\omega_2})^+ = \theta$. It follows that

$$\forall \alpha \in D_{M \cap \boldsymbol{H}_\theta} \quad f(\alpha) \in M \cap \boldsymbol{H}_\theta \subseteq M.$$

This, of course, is true for all f as described in the previous paragraph; so that $D_{M \cap \boldsymbol{H}_\theta} \subseteq D_M$. This implies unboundedness of D_M in ω_2. $\Box$

Fix a countable elementary submodel M of $\boldsymbol{H}_\lambda$, and let $M^{\{a\}}$ denote $sk_\lambda(M \cup \{a\})$.

Remark. $(\forall e \in D_M \setminus M)\ e = \min(M^{\{e\}} \cap \omega_2 \setminus M)$.

Proof. If $\delta < e$ and $\delta \in M^{\{e\}}$, then there is a formula ϕ with free variables in $M \cup \{e\}$ that uniquely defines δ. We define a regressive function $h \in M \cap {}^{\omega_2}\omega_2$ such that $h(e) = \delta$, which is indeed a regressive Skolem function, as follows:

$$h(z) = \begin{cases} \min\{\alpha < z\colon \phi(\alpha, a_1, \ldots, a_n, z)\} & \text{if not empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $h \in M \cap^{\omega_2} \omega_2$, and $h(e) = \delta$. It follows that $\delta \in M$. □

Similarly, we can show that for each $e \in D_M$ we have

$$M \cap \omega_1 = M^{\{e\}} \cap \omega_1.$$

3. A correspondence between $[\omega_2]^{\aleph_0}$ and $\mathcal{P}(\omega_1)$

Our basic tool in translating the structures between $[\omega_2]^{\aleph_0}$ and $\mathcal{P}(\omega_1)$ is a mapping Φ defined based on an *almost disjoint family* $\mathcal{A} = \{A_\alpha\colon \alpha < \omega_2\} \subset \mathcal{P}(\omega_1)$; that is for all distinct α and β in ω_2, $|A_\alpha \cap A_\beta| \leqslant \aleph_0$. The existence of such a family is a fact of ZFC. The map

$$\Phi : \big\langle [\omega_2]^{\aleph_0}, \subseteq \big\rangle \to \big\langle \mathcal{P}(\omega_1), \subseteq \big\rangle$$

first considered by Velickovic, is defined as follows:

$$\Phi(X) = \bigcup_{\alpha \in X} A_\alpha.$$

Notice that almost disjointness of $\mathcal{A}$ implies that

$$\big(\forall X \in [\omega_2]^{\aleph_0}\big)(\forall \alpha \notin X) \quad \Bigg| A_\alpha \setminus \bigcup_{\gamma \in X} A_\gamma \Bigg| = \aleph_1.$$

And, this implies that Φ is an order-preserving injection, hence an isomorphism onto its image. The range of Φ, however, is not all of $\mathcal{P}(\omega_1)$, for example, ω_1 is missing from $\Phi''([\omega_2]^{\aleph_0})$. Clearly, the image under Φ of any continuous strictly increasing chain of length ω_1 in $[\omega_2]^{\aleph_0}$ is a copy of ω_1 in $\mathcal{P}(\omega_1)$. The union of such a chain, however, does not belong to $[\omega_2]^{\aleph_0}$. For the purpose of finding converging ω_1 sequences in $\mathcal{P}(\omega_1)$ we will have to arrange for a copy of ω_1 in $[\omega_2]^{\aleph_0}$ whose image under Φ is destined to converge to a fixed point in $\mathcal{P}(\omega_1)$. If we plan to have a copy of ω_1 converging to an uncountable $Y \subseteq \omega_1$ we start with an almost disjoint family of subsets of Y.

In the next section we will be interested in finding homogeneous copies of ω_1 and of a *Baire space of weight* $\aleph_1$. The image under Φ of the following structure $\mathcal{Y}$ contains such spaces.

Construction of $\mathcal{Y}$ (RP *and* $2^{\aleph_0} = \aleph_2$). We start with a countable elementary submodel M. Therefore, $|{}^{<\omega_1}\omega_1| = \aleph_2$. Also, $|D_M| = \aleph_2$. We choose an ordering $\preceq$ of ${}^{<\omega_1}\omega_1$ in type ω_2 that respects the original partial ordering of ${}^{<\omega_1}\omega_1$.

Then, using Corollary 1, we construct $\mathcal{M} = \{M^s\colon s \in {}^{<\omega_1}\omega_1\}$ as follows:

(1) $M^\emptyset = M$,

(2) $M^s = \bigcup_{t \subset s} M^t$, for $s \in {}^{\alpha}\omega_1$, for any limit α,
(3) $M^{s^\frown\gamma} = sk_\lambda(M^s \cup \{e\})$, where

$$e = \min D_{M^s} \setminus \max\bigl\{\sup\bigl\{(M^t \cap \omega_2)\colon\ t \preceq s\bigr\}, \sup\bigl\{(M^{s^\frown\beta} \cap \omega_2)\colon\ \beta < \gamma\bigr\}\bigr\}.$$

For each $s \in {}^{<\omega_1}\omega_1$ let $X^s = M^s \cap \omega_2$, and $Y^s = \Phi(X^s)$, and define

$$\mathcal{Y} = \bigl\{Y^s\colon\ s \in {}^{<\omega_1}\omega_1\bigr\}.$$

Notice that each branch of $\mathcal{Y}$ is isomorphic to ω_1. Also, define

$$\mathcal{Y}' = \bigl\{Y^f\colon\ f \in {}^{\omega}\omega_1\bigr\} \quad \text{and} \quad Z = \Psi''(\mathcal{Y}'),$$

where $\Psi(X)$ is the characteristic function of X. As such, Z is a subset of 2^{ω_1} and with the subspace topology we have

Lemma 3. *Z is homeomorphic to $B(\omega_1)$.*

Proof. Recall that $B(\omega_1)$ is ${}^{\omega}\omega_1$ with the product topology, where ω_1 is given the discrete topology. Define $F : {}^{\omega}\omega_1 \to Z$, by $F(f) = \Psi(Y^f)$.

Claim 1. *F is a bijection.*

Proof. If $f \neq g$ then there is $n \in \omega$ such that $f(n) \neq g(n)$. Let $s = f|_n = g|_n$. Suppose $f(n) = \gamma$, $g(n) = \delta$, $s^\frown\gamma \preceq s^\frown\delta$, $M^{s^\frown\gamma} = sk_\lambda(M^s \cup \{\alpha\})$ and $M^{s^\frown\delta} = sk_\lambda(M^s \cup \{\beta\})$, for some α and β. By the construction, M^g will never contain α. Choose $\xi \in A_\alpha \setminus \Phi(X^g)$, and notice that $F(f)(\xi) \neq F(g)(\xi)$, so that $F(f) \neq F(g)$. $\square$

Claim 2. *F is open.*

Proof. Choose a subbasic open set of ${}^{\omega}\omega_1$, say

$$D(n, \alpha) = \bigl\{f \in {}^{\omega}\omega_1\colon\ f(n) = \alpha\bigr\}.$$

Observe that

$$\begin{aligned} F''\bigl(D(n,\alpha)\bigr) &= \bigl\{F(f)\colon\ f(n) = \alpha\bigr\} = \bigl\{\chi_{Y^f}\colon\ f(n) = \alpha\bigr\} \\ &= \bigl\{\chi_{Y^f}\colon\ f|_{n+1}(n) = \alpha\bigr\} = \bigl\{\chi_Y\colon\ Y^s \subset Y \text{ with } s(n) = \alpha\bigr\} \\ &= \bigl\{g \in Z\colon\ g(\gamma) = 1, \text{ where } \gamma \in Y^s, s(n) = \alpha\bigr\} \\ &= \bigcup_{s(n)=\alpha} \bigcup_{\gamma \in Y^s} = \bigl\{g \in Z\colon\ g(\gamma) = 1\bigr\}, \end{aligned}$$

which is open in the induced topology of Z from 2^{ω_1}. $\square$

Claim 3. *F is continuous.*

Proof. Choose a subbasic open set in Z, say for some $i \in \{0, 1\}$ and $\alpha < \omega_1$, $B_\alpha(i) = \{f \in Z\colon\ f(\alpha) = i\}$.

Case 1. $i = 1$.

$$B_\alpha(1) = \{h \in Z\colon h(\alpha) = 1\} = \{\chi_Y\colon Y \in \mathcal{Y}' \text{ and } \alpha \in Y\}.$$

Choose $s \in {}^{<\omega}\omega_1$ such that $\alpha \in Y^{s\frown\beta} \setminus Y^s$ for some $\beta \in \omega_1$. If no such s exists then either $\alpha \in Y^s$ for all s or $\alpha \notin Y^s$ for all s. Hence the set $\{Y \in \mathcal{Y}'\colon \alpha \in Y\}$ is either empty or it is the entire Z; so that $B_\alpha(1) \cap Z$, is empty, or equals to Z.

If such s exists then $B_\alpha(1) = \{\chi_{Yf}\colon s \subset f\}$. Then, $F^{-1}(B_\alpha(1)) = \{f \in {}^\omega\omega_1\colon s \subset f\}$ which is open is ${}^\omega\omega_1$.

Case 2. $i = 0$.

$$B_\alpha(0) = \{f \in Z\colon f(\alpha) = 0\} = \{\chi_Y\colon Y \in \mathcal{Y}' \text{ and } \alpha \notin Y\}.$$

Consider s as before. If no such s exists then either $\alpha \notin Y^s$ for all s or $\alpha \in Y^s$ for all s. In the former case $F^{-1}(B_\alpha(0)) = Z$, and in the latter $F^{-1}(B_\alpha(0)) = \emptyset$, both open sets.

If such a s exists then for some β

$$B_\alpha(0) = \{\chi_{Yf}\colon s \subset f \text{ and } s\frown\beta \not\subset f\} \cup \{\chi_{Yf}\colon s \subset f\}^c,$$

which is open because each component of the union is clopen. □

This establishes the continuity of F, and finishes the proof. □

4. SPFA

A *semi-proper* partial order is a forcing notion which does not destroy stationary subsets of ω_1. *The Semi-Proper Forcing Axiom* (SPFA) is the following statement:

Semi-Proper Forcing Axiom. *For any collection $\mathcal{D}$ of dense open subsets of a semi-proper partial ordering $\mathbb{P}$ there exists a $\mathcal{D}$-generic filter G, i.e., a $G \subseteq \mathbb{P}$ such that*:

(1) *G is a filter, and*
(2) *for all $D \in \mathcal{D}$, $G \cap D \neq \emptyset$.*

A typical use of SPFA that we shall have here is as follows. Suppose the elements of some partial order $\mathbb{P}$ are structures similar to countable ordinals, and a suitable $\mathcal{D}$ guarantees the extension of each element of $\mathbb{P}$ into arbitrary large countable structures. Then, the $\mathcal{D}$-generic filter G will contain large enough “compatible” structures, so that $\bigcup G$ becomes an structure similar to ω_1.

It is known (see Bekkali [1]) that SPFA implies RP and $2^{\aleph_0} = \aleph_2$, hence the construction in Section 3 is a consequence of SPFA.

Proposition 1 (SPFA). *The following topological partition relations hold*:

(a) $2^{\omega_1} \to (\omega_1 + 1, \omega_1)^1$,
(b) $2^{\omega_1} \to (\omega_1 + 1, B(\omega_1))^1$.

Proof. It would be sufficient to prove that

$$\mathcal{P}(\omega_1) \to (\omega_1 + 1, \mathcal{Y})^1,$$

where $\mathcal{Y}$ is as constructed in Section 3.

Given a partition $\{X_0, X_1\}$ of $\mathcal{P}(\omega_1)$ we assume that $\mathcal{Y} \not\hookrightarrow X_1$. Observe that X_0 contains an uncountable subset of ω_1, since otherwise for any uncountable co-uncountable subset Y of ω_1, we would have

$$[Y, \omega_1] = \{Z\colon\ Y \subseteq Z \subseteq \omega_1\} \subseteq X_1.$$

Notice that $[Y, \omega_1]$ as a lattice is isomorphic to $\mathcal{P}(\omega_1)$, which implies $\mathcal{Y} \hookrightarrow X_1$; a contradiction.

Choose an uncountable $Y \in X_0$, and let $\mathcal{A} = \{A_\alpha\colon\ \alpha < \omega_2\}$ be an almost disjoint family of subsets of Y such that $Y = \bigcup \mathcal{A}$. As in Section 3 we define a map Φ and we let $K_i = \Phi^{-1}(X_i)$, for $i = 0, 1$. Then we define $\mathbb{P}$ as follows:

$p \in \mathbb{P}$ iff $p = \langle X_\alpha\colon\ \alpha \leqslant \gamma\rangle \subseteq K_0$ is a countable continuous increasing closed chain. Notice that each condition p is considered to be a function with domain $\gamma + 1$ for some countable ordinal γ. Similarly, *range*(p) is the collection $\{X_\alpha\colon\ \alpha \leqslant \gamma\}$ of countable subsets of ω_2. The ordering of $\mathbb{P}$ is end-extension.

Case 1. $\mathbb{P}$ is *semi-proper*.

Define for each $\alpha \in \omega_1$

$$D_\alpha = \{p \in \mathbb{P}\colon\ \alpha \in \mathit{dom}\, p\},$$

and for each $\alpha \in Y$, set

$$E_\alpha = \bigl\{p \in \mathbb{P}\colon\ (\exists B \in \mathit{range}(p))(\exists \beta \in B)\ \alpha \in A_\beta\bigr\}.$$

We will argue that D_α and E_α's are dense in $\mathbb{P}$. D_α's are dense, as otherwise any extension of some $\{X_\xi\colon\ \xi \leqslant \gamma\} \subseteq [\omega_2]^{\aleph_0}$ is in K_1. This would mean that $[\omega_2]^{\aleph_0} \hookrightarrow K_1$, which implies $\mathcal{Y} \hookrightarrow X_1$. Similarly, E_α's are dense because

$$(\forall \alpha \in Y)(\exists \beta \in \omega_2)\ \alpha \in A_\beta,$$

and for each $\beta \in \omega_2$, the collection $\mathcal{B} = \{B \in [\omega_2]^{\aleph_0}\colon\ \beta \in B\}$ is isomorphic to $[\omega_2]^{\aleph_0}$ and therefore must contain element from K_0 extending any given chain $p = \{X_\alpha\colon\ \alpha \leqslant \gamma\}$ from $\mathcal{P}$.

We define

$$\mathcal{D} = \{D_\alpha\colon\ \alpha \in \omega_1\} \quad \text{and} \quad \mathcal{E} = \{E_\alpha\colon\ \alpha \in Y\},$$

and by semi-properness of $\mathbb{P}$ we choose a $\mathcal{D} \cup \mathcal{E}$-generic subset $\boldsymbol{G}$ of $\mathbb{P}$.

Observe that $g = \bigcup \boldsymbol{G}$ satisfies:

(1) $g : \omega_1 \to K_0$ is a continuous, increasing map, hence $\{\Phi(g(\alpha))\colon\ \alpha < \omega_1\}$ is homeomorphic to ω_1, and is contained in X_0;

(2) by the $\mathcal{E}$-genericity of G we have $Y = \bigcup_{\alpha \in \omega_1} \Phi(g(\alpha))$.

Of course, (2) implies that the copy of ω_1 created by (1) will converge to Y (i.e., its union is Y); which gives rise to a copy of $\omega_1 + 1$ in X_0.

Case 2. $\mathbb{P}$ is not *semi-proper*. Then, there must exist a stationary subset S of ω_1 such that,

$$\Vdash_{\mathbb{P}} \text{“}\check{S} \text{ is not stationary in } \check{\omega}_1\text{”}.$$

Fix a name τ, a stationary S as above, and a condition p such that:

$$p \Vdash \text{“}\tau \text{ is a club in } \check{\omega}_1, \text{ and } \check{S} \cap \tau = \emptyset\text{”}.$$

Let $\lambda = (2^{2^{\aleph_2}})^+$ and notice that $\boldsymbol{H}_\lambda$ contains all the information discussed this far. Take a countable elementary $M \prec \boldsymbol{H}_\lambda$ which contains the above information as well as $M \cap \omega_1 = \delta \in S$.

Subcase 1. For all the elementary end-extensions N of M we have $N \cap \omega_2 \in K_1$. Recall that in Section 3.3 we constructed a tree $\mathcal{M} = \{M^s\colon s \in {}^{<\omega_1}\omega\}$ of elementary end-extensions of a given elementary submodels and a tree $\mathcal{Y}$ of the traces of elements of $\mathcal{M}$ on ω_2. Each branch of this tree translates to a copy of ω_1 and an initial part of this tree translates to $B(\omega_1)$ (see Section 3.3). Indeed the assumption that for any end-extension N of M we have $N \cap \omega_2 \in K_1$, implies that $\mathcal{Y} \subset K_1$ which is more than needed in both cases (a) and (b) of proposition.

Subcase 2. For some end-extension N of M, we have $N \cap \omega_2 \in K_0$.

Claim. *There is a descending chain of conditions $\{p_n\colon n < \omega\}$ below p, and an increasing sequence of countable ordinals $\{\gamma_n\colon n < \omega\}$ cofinal in δ such that*

(1) $p_n \Vdash \text{“}\check{\gamma}_n \in \tau\text{”}$,

(2) $\bigcup_{n\in\omega} \mathit{dom}\, p_n = \delta$, *and*

(3) $\bigcup_{n\in\omega}(\bigcup \mathit{range}\, p_n) = N \cap \omega_2$.

Proof. Pick a sequence $\langle \delta_n\colon n \in \omega\rangle$ of ordinals less than δ converging to δ, let $N \cap \omega_2 = \{\zeta_n\colon n \in \omega\}$, and define

$$D_n = \Big\{q \in \mathbb{P}\colon \zeta_n \in \bigcup \mathit{range}\, q \text{ and } \delta_n \in \mathit{dom}\, q\Big\}.$$

Clearly, for each $n \in \omega$, $D_n \in N$ and D_n is dense in $\mathbb{P}$. Since

$$N \models \text{“}p \Vdash \text{“}\tau \text{ is a club in } \check{\omega}_1\text{””},$$

$$N \models \text{“}(\exists \gamma_0 \geqslant \delta_0)(\exists p_0 \in D_0)\ p_0 \leqslant p\ p_0 \Vdash \text{“}\check{\gamma}_0 \in \tau\text{””}.$$

Next,

$$N \models \text{“}\big(\exists \gamma_1 \geqslant \max(\gamma_0, \delta_1)(\exists p_1 \in D_1)\ p_1 \leqslant p_0 \text{ and } p_1 \Vdash \text{“}\check{\gamma}_1 \in \tau\text{””},$$

and so on. Inductively, a descending sequence $\{p_n\colon n < \omega\}$ is constructed, such that (1) holds and for each $n \in \omega$, $p_n \in D_n \cap N$. Then,

$$\bigcup_{n<\omega} \mathit{dom}\, p_n = \delta \quad \text{and} \quad \bigcup_{n\in\omega}\bigcup \mathit{range}\, p_n = N \cap \omega_2. \quad \square$$

Finally,

$$q = \bigcup_{n\in\omega} p_n \cup \big\{\langle \delta, M \cap \omega_2\rangle\big\},$$

is a condition, as it is continuous and closed. Notice that

$$q \Vdash \text{“}\{\check{\gamma}_n\colon n < \check{\omega}\} \subset \tau \text{ and is unbounded below } \delta\text{”}.$$

Therefore, $q \Vdash \text{“}\check{\delta} \in \tau\text{”}$.

But, $q \leqslant p$ and $p \Vdash$ "$\check{\delta} \in \check{S}$". It follows that $q \Vdash$ "$\check{\delta} \in \tau \cap \check{S}$", which contradicts $p \Vdash$ "$\tau \cap \check{S} = \emptyset$". $\square$

5. Chang's Conjecture

The SPFA used in the last section is a forcing axiom whose consistency proof uses a large cardinal assumption. Would the consistency proof of $[\omega_2]^{\aleph_0} \to (\omega_1)^1_2$ indeed need a large cardinal hypothesis? In this section we will demonstrate that this is indeed the case (see corollary of Proposition 2).

Chang's Conjecture (hereafter CC). *Every structure of the form $\langle \omega_2, \omega_1, <, \ldots \rangle$ for a countable language has an uncountable elementary submodel B such that $B \cap \omega_1$ is countable.*

It is known (see [4]) that CC implies $0^\sharp$, which is a large cardinal assumption.

Choose a counter-example $\mathfrak{A} = \langle \omega_2, \omega_1, h, <, \ldots \rangle$ to CC that contains as a predicate a function $h : [\omega_2]^2 \to \omega_1$ with the property that $h(\alpha, \gamma) \in \omega$ and $h(\alpha, \gamma) \neq h(\beta, \gamma)$ $\alpha < \beta < \gamma$, and $\alpha, \gamma \in \omega_1$. To each $\{\alpha, \beta\}$, we associate

$$B_{\alpha\beta} = sk_{\mathfrak{A}}\big(\{\alpha, \beta\}\big),$$

where, of course, $sk_{\mathfrak{A}}(\{\alpha, \beta\})$ is Skolem closure of $\{\alpha, \beta\}$ in the structure $\mathfrak{A}$, that is the smallest elementary model of $\mathfrak{A}$ that contains α and β. Notice that $B_{\alpha\beta} \cap \omega_1$ is a countable ordinal, so we can define the function $e : [\omega_2]^2 \to \omega_1$ as follows:

$$e(\alpha, \beta) = B_{\alpha\beta} \cap \omega_1.$$

It is easily seen that (see, for example, Todorcevic [7])

Lemma 4. *For every uncountable $A \subseteq \omega_2$, the image $e''([A]^2)$ is uncountable.*

Notation. For a topological space X and a point $x_0 \in X$ let

$$\sum_{\omega_2}^{x_0} X = \big\{ f \in {}^{\omega_2}X \colon\ |\operatorname{supp}(f)| \leqslant \aleph_0 \big\},$$

where, $\operatorname{supp}(f) = \{\alpha \in \omega_2 \colon\ f(\alpha) \neq x_0\}$. This is the Σ-product of the topological space X around the point x_0 (see Engelking [2]).

Proposition 2. *Suppose X is a topological space such that $X^{\aleph_0}$ contains no copy of ω_1. Then, $\sum_{\omega_2}^{x_0} X \to (\omega_1)^1_2$ implies* CC.

Proof. Assume the negation of CC and consider the function e as above. Consider a partition $\{E_0, E_1\}$ of ω_1 into stationary co-stationary subsets, and define $\Phi : \sum_{\omega_2}^{x_0} X \to \{0, 1\}$ as follows: for a given $g \in \sum_{\omega_2}^{x_0} X$,

$$\Phi(g) = i \quad \text{iff} \quad \sup\big(e''[\operatorname{supp}(g)]^2\big) \in E_i, \quad i = 0, 1.$$

Notations.

(1) $\mathcal{F}=\{f_\alpha\colon \alpha<\omega_1\}$ denotes any copy of ω_1 in $\sum_{\omega_2} X$.

(2) $S_\alpha=\operatorname{supp}(f_\alpha)$, $\mathcal{S}=\{S_\alpha\colon \alpha<\omega_1\}$ and for each $\alpha<\omega_1$, $\lambda_\alpha=\sup(e''[S_\alpha]^2)$.

(3) For each $\alpha\in\omega_2$, $I_\alpha=\{\gamma\in\omega_1\colon \alpha\in S_\gamma\}$.

(4) $A=\{\alpha\in\omega_2\colon I_\alpha$ is uncountable$\}$.

(5) $\langle M_\xi\colon \xi<\omega_1\rangle$ is an $\in$-chain of elementary submodels of $\boldsymbol{H}_\lambda$ for some large enough λ such that $\boldsymbol{H}_\lambda$ contains all the information discussed so far. Also assume $e,\mathcal{F},X,\Phi,\mathcal{S},A$ all belong to M_0.

(6) C is a club in ω_1 such that $\forall\delta\in C$ $M_\delta\cap\omega_1=\delta$.

Claim 1. $(\forall\delta\in C)$ $S_\delta\subseteq M_\delta$.

Proof. For all $\alpha<\delta$, $S_\alpha\in M_\delta$; since $\{f_\xi\colon \xi<\omega_1\}\in M$ hence $S_\alpha\subseteq M_\delta$ as S_α is countable. By continuity at δ, every $\beta\in S_\delta$ must belong to S_α for cofinally many α below δ; which implies that $\beta\in M_\delta$. $\square$

Claim 2. $(\forall\alpha<\omega_1)$ $(\exists\beta>\alpha)$ $S_\beta\setminus D_\alpha\neq\emptyset$.

Proof. Otherwise, the support of $\mathcal{F}_\alpha=\{f_\beta\colon \beta>\alpha\}$ is contained in S_α, which is a countable set, while $\mathcal{F}_\alpha$ is homeomorphic to ω_1. This means that the countable product X^{S_α} contains a copy of ω_1, which contradicts our assumption about X. $\square$

Remark 1. Notice that if C is a club in ω_1, then $\{f_\alpha\colon \alpha\in C\}$ is also homeomorphic to ω_1 so that the above argument produces an ordinal $\beta\in C$ as described above.

Lemma 5. $(\forall\delta\in C)$ $S_\delta=A\cap M_\delta$.

Proof. $(\subseteq)$ Choose $\delta\in C$ and $\alpha\in S_\delta$. Then, $\alpha\in M_\delta$ as $S_\delta\subseteq M_\delta$, by Remark 1. Assume for a moment that $\alpha\notin A$. Then, as $A\in M_0\subset M_\delta$,

$$M_\delta\models\alpha\notin A.$$

Hence,

$$M_\delta\models I_\alpha \text{ is countable.}$$

It follows that $I_\alpha\subseteq M_\delta\cap\omega_1=\delta$, so $\delta\notin I_\alpha$, a contradiction.

$(\supseteq)$ Choose $\alpha\in A\cap M_\delta$. Then $M_\delta\models\alpha\in A$, so that

$$M_\delta\models \text{“}I_\alpha \text{ is uncountable”}.$$

This translates to

$$(\forall\beta<\delta)\ (\exists\eta<\delta)\beta<\eta \quad\text{and}\quad \alpha\in S_\eta,$$

and together with the fact that “$\mathcal{F}$ is continuous at δ” it gives that $\alpha\in S_\delta$. This finishes the proof. $\square$

Claim 1. *A is uncountable.*

Proof. Fix a $\delta \in C$. Since $A \in M_\delta$, then if it were countable it had to be a subset of M_δ. Hence, to prove A is uncountable it is enough to prove that $A \setminus M_\delta \neq \emptyset$. To this end, choose a $\gamma > \delta$ in C such that $S_\gamma \setminus S_\delta \neq \emptyset$. The existence of such a $\gamma \in C$ is guaranteed by Claim 2 and the comment following it. Choose $\alpha \in S_\gamma \setminus S_\delta$. Because $S_\gamma = A \cap M_\gamma$ and $S_\delta = A \cap M_\delta$, we have $\alpha \in A \setminus M_\delta$. This finishes the proof. $\square$

Recall that $\lambda_\alpha = \sup(e''[S_\alpha]^2)$. We are planning to prove that $\Lambda = \{\lambda_\alpha\colon \alpha \in C\}$ contains a *club* subset of ω_1; so that

$$\Lambda \cap E_i \neq \emptyset \quad \text{for every } i = 0, 1,$$

which implies

$$\Phi''(\mathcal{F}) = \{0, 1\}.$$

Since $\mathcal{F}$ has an arbitrary copy of ω_1, this means that Φ witnesses $\Sigma^{x_0}_{\omega_2} X \nrightarrow (\omega_1)^1_2$, finishing the proof of Proposition 2.

Claim 2. *For all $\delta \in C$, $\lambda_\delta = \delta$.*

Proof. ($\Leftarrow$) For all α and β in S_δ we have $e(\alpha, \beta) \in M_\delta$ because $S_\delta \subseteq M_\delta$. Hence, $e(\alpha, \beta) \leqslant \delta$, so that $\lambda_\delta \leqslant \delta$.

($\Rightarrow$) Suppose $\lambda_\delta < \delta$, then $\lambda_\delta \in M_\delta$. Now it follows from Lemma 4 that

$$M_\delta \models \text{“}(\exists \alpha, \beta \in A)\ e(\alpha, \beta) > \lambda_\delta\text{”},$$

that is,

$$(\exists \alpha, \beta \in M_\delta \cap A)\ e(\alpha, \beta) > \lambda_\delta.$$

By Lemma 5 we have that $\alpha, \beta \in S_\alpha$, which contradicts $\lambda_\delta = \sup(e''[S_\delta]^2)$. It follows from Claim 2 that $C \subseteq \Lambda$ and this finishes the proof. $\square$

Corollary 2. $\Sigma^0_{\omega_2} \to (\omega_1)^1_2$ *implies* CC. *Or more precisely,* $[\omega_2]^{\aleph_0} \to (\omega_1)^1_2$ *implies* CC.

6. Partitioning the Σ-product into three pieces

In this section we modify the proof of Proposition 2 to establish the following:

Proposition 3. *Suppose X is a topological space such that $X^{\aleph_0}$ contains no copy of ω_1. Then,*

$$\sum_{\omega_2}^{x_0} X \nrightarrow (\omega_1)^1_3.$$

With the notation as in the previous section for each $g \in \sum^{x_0}_{\omega_2} X$, we define

$$D(g) = sk_{\mathfrak{A}}\big(\mathrm{supp}(g)\big).$$

Observe that:

(1) For each g, $D(g) \cap \omega_1$ is an ordinal.

(2) If $\mathcal{G} = \{g_\alpha\colon \alpha < \omega_1\}$ is a sequence of elements of $\Sigma_{\omega_2}^{x_0} X$ such that $\langle \mathrm{supp}(g_\alpha)\colon \alpha < \omega_1\rangle$ is *increasing* and *continuous* (that is for $\alpha < \beta$, $\mathrm{supp}(g_\alpha) \subseteq \mathrm{supp}(g_\beta)$ and $\mathrm{supp}(g_\alpha) = \bigcup_{\gamma<\alpha} \mathrm{supp}(g_\gamma)$ for limit α), then $\{D(g_\alpha) \cap \omega_1\colon \alpha < \omega_1\}$ is an increasing continuous collection of ordinals in ω_1.

(3) For each g and h with $\mathrm{supp}(g) \subseteq \mathrm{supp}(h)$

$$D(g) \cap \omega_1 = D(h) \cap \omega_1 \Rightarrow \min\bigl(D(h) \setminus D(g)\bigr) \cap \omega_2 \geqslant \sup\bigl(D(g) \cap \omega_2\bigr).$$

Proof. Let $\alpha \in (D(h) \cap \sup D(g)) \cap \omega_2$. We claim that $\alpha \in D(g)$.

Choose $\gamma \in D(g) \cap \omega_2$ such that $\gamma > \alpha$ and consider the ordinal

$$\xi = f(\alpha, \gamma) \in D(h) \cap \omega_1 = D(g) \cap \omega_1.$$

Since $D(g)$ is an elementary submodel of $(\omega_2, \omega_1, f, \ldots)$ which contains ξ and γ and since α is the unique solution to the equation $\xi = f(\alpha, \gamma)$ if follows that $\alpha \in D(g)$. $\square$

Proof. We define maps

$$\Phi : \sum_{\omega_2}^{x_0} X \to \omega_1, \quad \Phi(g) = D(g) \cap \omega_1 \quad \text{and}$$

$$\Psi : \sum_{\omega_2}^{x_0} X \to \omega_1, \quad \Psi(g) = otp(D(g) \cap \omega_2).$$

Claim. *Suppose the supports of the elements in $\mathcal{G} = \{g_\alpha\colon \alpha < \omega_1\}$ form an increasing and continuous chain of sets, then at least one of the $\Phi''\mathcal{G}$ or $\Psi''\mathcal{G}$ contains a club.*

Proof. It follows from observation (2) that $\Phi''\mathcal{F}$ is increasing and continuous, as the supports are increasing and continuous. If it is unbounded, $\Phi''\mathcal{F}$ will be a *club* in ω_1, and if it is bounded then there must be some α_0 such that

$$\Phi(g_\beta) = \Phi(g_{\alpha_0}) \quad \forall \beta \geqslant \alpha_0.$$

In this case, by observation (3) we have $D(g_\beta)$ end-extends $D(g_\alpha)$ whenever $\beta > \alpha \geqslant \alpha_0$. This property of the $D(g_\alpha)$'s will guarantee that $\langle \Psi(g_\alpha)\colon\ (\alpha_0 \leqslant \alpha < \omega_1)\rangle$, form an increasing continuous collection of ordinals in ω_1 of length ω_1. It follows that in this case $\Psi''\mathcal{G}$ contains a club subset of ω_1. $\square$

Now we define a coloring $c : \sum_{\omega_2}^{x_0}(X) \to \{0, 1, 2\}$ as follows: let $\{S_0, S_1, S_2\}$ be a partition of ω_1 into stationary sets, then

$$c(g) = \min\bigl\{i\colon\ \Phi(g) \notin S_i \text{ and } \Psi(g) \notin S_i\bigr\}.$$

To show that this partition does not allow a homogeneous copy of ω_1 we assume $\mathcal{F} = \{f_\alpha\colon \alpha < \omega_1\}$ is any copy of ω_1 in $\sum_{\omega_2}^{x_0} X$. Using the arguments in the previous section we can select a subcollection $\mathcal{G} = \{g_\alpha\colon \alpha < \omega_1\}$ of $\mathcal{F}$ such that the supports $\{\mathrm{supp}(g_\alpha)\colon \alpha < \omega_1\}$ form an increasing continuous chain of subsets of ω_2.

It follows from the claim that one of the $\Phi''\mathcal{G}$ or $\Psi''\mathcal{G}$ will be a club, hence intersecting all the S_i's. That is, either for some g_0, g_1 and g_2 in $\mathcal{G}$, we have $\Phi(g_i) \in S_i$ for $i = 0, 1, 2$, or that for some g_1, g_2 and g_2 in $\mathcal{G}$, $\Psi(g_i) \in S_i$ for $i = 0, 1, 2$. In any case we have

$$c(g_0) \neq 0, \quad c(g_1) \neq 1 \quad \text{and} \quad c(g_2) \neq 2.$$

This means that $\mathcal{F}$ is colored by c in more than one color. $\square$

Remark. It follows from Proposition 3 that the negative result above is strongest possible, in the sense that the number of colors cannot be decreased to two.

7. Negative consistency results: Generic filter as a partition

In this section we force with a countably closed partial order. The generic filter in this partial order will split a given topological space into two, with no homogeneouscopy of an uncountable first topological space.

First we bring a definition and a lemma from general topology.

Definition 2. A topological space X is said to concentrate about $A \subset X$ whenever any open set about A contains all but at most countably many points of X.

Lemma 6. *Uncountable first countable countably compact Hausdorff spaces cannot be concentrated about their countable subsets.*

Proof. First observe that:

(1) Any topological space concentrated about a countable set is Lindelöf, hence it is enough to prove: "no uncountable, first countable compact space is concentrated about countable subsets".

(2) No uncountable, first countable space can be concentrated about one point.

(3) Let $B = \{x \in X\colon \ \text{for any open} \ U \ x \in U \Rightarrow |U \cap X| > \aleph_0\}$. Then, B is closed. Also, if X is compact first countable then B is dense in itself. This can be justified as follows. Let $x_0 \in B$ be isolated and find open set U such that $U \cap B = \{x_0\}$. Fix a countable decreasing collection of open sets $\{U_n\colon n \in \omega\}$ with $\overline{U_0} \subseteq U$ and $\bigcap_{n\in\omega} U_n = \{x_0\}$. Since $U_0 \setminus \{x_0\} = \bigcup_{n\in\omega}(U_n \setminus U_{n+1})$ is uncountable for some n_0, $U_{n_0} \setminus U_{n_0+1}$ must be uncountable. Let y be a complete accumulation point of $U_{n_0} \setminus U_{n_0+1}$. Then, $y \in \overline{U_0} \setminus U_{n_0+1} \subset U$. But, by definition, $y \in B$; a contradiction.

(4) If $B \neq \emptyset$ and X is compact first countable, then B is uncountable. This is an immediate consequence of Čech–Pospíšil's lemma (see Engelking [2]) and of the previous observation on B.

(5) If X is uncountable and concentrated about $A = \{a_n\colon n \in \omega\}$, then $B \cap A \neq \emptyset$ (in particular $B \neq \emptyset$). To see this, suppose $A \cap B = \emptyset$ and choose countable open neighborhoods U_n of a_n. But, U is countable, $A \subset U = \bigcup_{n\in\omega} U_n$, and as U is open, $X \setminus U$ is countable; a contradiction.

To prove the lemma, we assume X is an uncountable, compact, first countable space concentrated about a set $A = \{a_n\colon\ n \in \omega\}$. Consider B as in observation (3). Notice that by observations (4) and (5), B is first countable, compact dense in itself. Hence, we may assume without loss of generality that X has these properties.

Aiming towards a contradiction, we will find an open set $U \supset A$ whose complement is uncountable.

We use (2) to find an open set $U_0 \ni a_0$ whose complement in uncountable. Then, we choose an uncountable open set $V_{\langle 0\rangle}$ whose closure is disjoint from closure of U_0. Next, we apply (2) again to find an open set $U_1 \ni a_1$ together with uncountable open sets $V_{\langle 0,i\rangle}$, $i = 0, 1$ such that:

(1) $\overline{V_{\langle 0,i\rangle}} \subset V_{\langle 0\rangle}$,

(2) $\overline{V_{\langle 0,0\rangle}} \cap \overline{V_{\langle 0,1\rangle}} = \emptyset$, and

(3) $\overline{V_{\langle 0,i\rangle}} \cap \overline{U_1} = \emptyset$.

Having constructed open sets $U_n \ni a_n$, $n \in \omega$, and $\mathcal{V} = \{V_s\colon\ s \in {}^{<\omega}2\}$ such that

(1) $\overline{V_s} \subset V_t$, whenever s extends t,

(2) $\overline{V_s} \cap \overline{U_n} = \emptyset$, for all n and all $s \in {}^n2$, and

(3) $\overline{V_{s^\frown 0}} \cap \overline{V_{s^\frown 1}} = \emptyset$,

we define

$$Z = \bigcup_{f \in {}^\omega 2} \bigcap_{n \in \omega} V_{f|_n}.$$

Let $U = \bigcup_{n\in\omega} U_n$ and notice that we have:

(1) $Z \cap U = \emptyset$,

(2) $A \subset U$,

(3) Z is uncountable.

Thus, U is the desired open set. $\square$

Given a cardinal κ and a topological space $X = \langle \kappa, \mathcal{T}\rangle$, over a model of ZFC + CH we force with $\mathbb{P} = Fn(\kappa, 2, \omega_1)$. For any generic filter G, $\bigcup G : \kappa \to 2$ is a partition of κ, hence a partition of X into two pieces.

Proposition 4. *With respect to the above generic partition, there is no homogeneous copy of a countably compact uncountable first countable space inside X.*

Before proving the proposition we mention a basic consequence of countable closedness of $\mathbb{P}$:

Remark 2. With the above notation, if A is countable, and ζ is a name such that $p \Vdash$ "ζ is a subspace of $\check{X}$ with $\check{A} \subset \zeta$, and $(\forall a \in \check{A})\ \chi(a, \zeta) = \check{\aleph}_0$", then:

(1) There is a condition $q \leqslant p$ and there is a countable collection of open sets $\mathcal{W}_A$ such that for any name ν:

$$q \Vdash \text{“}\nu \text{ is open and } (\forall a \in \check{A})\ a \in \nu \Rightarrow (\exists W \in \mathcal{W}_A)\ a \in W \cap \zeta \subset \nu \cap \zeta\text{”}.$$

(2) There is a condition $q \leqslant p$ such that for any name ν,

$$q \Vdash \text{“}\nu \text{ is open in } \zeta \text{ and } \check{A} \subset \nu \Rightarrow (\exists W \in \check{\mathcal{T}})\check{A} \subset W \cap \zeta \subset \nu\text{”}.$$

Proof.

(1) Let $A = \{a_n\colon\ n \in \omega\}$. There are names ν_n for $n \in \omega$, such that,

$$p \Vdash \text{“}\nu_n \text{ is a countable local basis at } \check{a}_n \text{ with respect to } \zeta\text{”}.$$

Since $\mathcal{T}$ is a basis for the topology of X in the extension, one can find a condition $p_0 \leqslant p$ and a countable collection, $\mathcal{W}_0$, of open sets such that

$$p_0 \Vdash \text{“}(\forall \text{ open } \nu \ni \check{a}_0)(\exists W \in \check{\mathcal{W}}_0)\ \check{a} \in W \cap \zeta \subset \nu\text{”}.$$

We call this process *deciding a local basis for a_0, below p*.

Next, we decide a local basis for a_1, below p_0, and continue to construct a descending chain of conditions p_n, $n \in \omega$, along with countable collections of open sets $\mathcal{W}_n$ which are decided by the p_n's to be the ground model local basis for the a_n's.

The desired collection $\mathcal{W}_A$ is the union of $\mathcal{W}_n$'s and the desired condition is $q = \bigcup_{n<\omega} p_n$.

(2) In this part we use the condition q and $\mathcal{W}_A$ as above. Given a name ν, such that $q \Vdash$ “$A \subset \nu$ and ν is open”, we know from (1) that $q \Vdash$ “$(\forall n \in \check{\omega})(\exists W_n \in \check{\mathcal{W}}_n)\ a_n \in W_n \cap \zeta \subset \nu$”, i.e.,

$$q \Vdash \text{“}\exists W\left(= \bigcup_{n\in\omega} W_n\right)\ \check{A} \subset W \cap \zeta \subset \nu\text{”}.$$

But $W \in \mathcal{T}$ as it is a countable union of elements of $\mathcal{T}$ and our partial order is σ-complete. $\square$

Proof of Proposition 4. Fix names ξ and τ, and a condition $p \in \mathbb{P}$ such that $p \Vdash \xi$ is an uncountable first countable countably compact, and $\tau : \xi \hookrightarrow \bigcup \Gamma^{-1}(0)$.

Define for $r \leqslant p$,

$$A^r = \left\{x \in dom(r)\colon\ (\exists q \leqslant r) q \Vdash \text{“}(\exists \alpha \in \xi)\ \tau(\alpha) = \check{x}\text{”}\right\}$$

and

$$A_r = \left\{x \in X\colon\ r \Vdash \text{“}(\exists \alpha \in \xi)\ \tau(\alpha) = \check{x}\text{”}\right\}.$$

Claim 1. *With the notation as above, for any condition r there exists $q \leqslant r$ such that $q \Vdash$ “$\check{A}^q \subset \tau''(\xi)$”.*

Proof. Let $q_0 = r$. For each positive integer n, having chosen q_n, as $dom\, q_n$ is countable, by countable closedness we choose a condition $q_{n+1} \leqslant q_n$ such that $q_{n+1} \Vdash$ “$A^{q_n} \subseteq \tau''(\xi)$”. Define $q = q_\omega = \inf\{q_n\colon\ n \in \omega\}$ and notice that $A^q = \bigcup_{n\in\omega} A^{q_n}$, and $q \Vdash$ “$A^q \subseteq \tau''(\xi)$”. $\square$

We are planning to find a $z \in A$ and a condition below p which contains $\langle z, 1\rangle$. This, of course, implies that the image of τ meets the color 1 as well; hence the desired contradiction is reached.

Notice that whenever, in the following construction, for some $r \leqslant p$ it happens that $A_r \setminus dom\, r \neq \emptyset$, then we may choose $z \in A_r \setminus dom\, r$ and extend r to $r \cup \{\langle z, 1\rangle\}$ to get the required condition. Thus, we assume throughout, that for any $r \leqslant p$ we have $A_r \subseteq dom\, r$.

Claim 2. *Given a condition $q \leqslant p$ there exists $r \leqslant q$ and a $\mathcal{W}_r \subset \mathcal{T}$ such that $r \Vdash$ "ν is open in $\tau''(\xi)$ and $\check{A}^r \subset \nu'' \Rightarrow (\exists W \in \check{\mathcal{W}}_r)\ \check{A}^r \subset W \cap \tau''(\xi) \subset \nu$".*

Proof. By Claim 1 extending q we may assume that $q \Vdash$ "$\check{A}^q \subseteq \check{\tau}(\xi)$". We use Remark 2 to find a condition $q_0 \leqslant q$ and a ground model local basis $\mathcal{W}_0$ for A^q in the subspace $\tau''(\xi)$. We repeat to find $q_{n+1} \leqslant q_n$ and a ground model local basis $\mathcal{W}_{n+1}$ for A^{q_n}. We let $r = \bigcup_{n<\omega} q_n$ and $\mathcal{W}_r = \bigcup_{n<\omega} \mathcal{W}_n$. $\square$

Claim 3. *There exist*:

(1) *a sequence $\{\eta_n\colon n < \omega\}$ of elements of ξ and a countable descending chain of conditions $p_n \leqslant p$, $n < \omega$,*

(2) *a sequence of open sets $\{U_n\colon n < \omega\}$ such that $A^{p_n} \subset U_n$, and*

(3) *a countable collection of points $\langle x_n\colon n < \omega\rangle \subset X$ such that $x_{n+1} \notin U_n$ for $n < m$ and $p_n \Vdash$ "$\tau(\eta_n) = \check{x}_n$".*

Proof. Fix a name η_0, and find $q_0 \leqslant p$ and $x_0 \in X$ to satisfy

$$q_0 \Vdash \text{“}\eta \in \tau''(\xi) \text{ and } \tau(\eta_0) = \check{x}_0 \text{ and } A^{q_0} \subset \tau''(\xi)\text{”}.$$

Since $q_0 \Vdash$ "$\tau''(\xi)$ is first countable countably compact", then by Proposition 1, $q_0 \Vdash$ "($\exists U$ open in $\tau''(\xi))\check{A}^q \subset U$ and $\tau''(\xi) \setminus U$ is uncountable". Claim 2 guarantees that there is a condition $p_0 \leqslant q_0$ and an open set U_0 with $A^{p_0} \subset U_0$ and $p_0 \Vdash$ "$\check{U}_0 \subseteq U$". It follows that $p_0 \Vdash$ "$\tau''(\xi) \setminus U_0$ is uncountable". Choose another name η_1, another point $x_1 \notin A \setminus U_0$ and $q_1 \leqslant p_0$ such that $q_1 \Vdash$ "$\tau(\eta_1) = \check{x}_1$". Then, we apply Claim 1 to find p_1 such that $p_1 \Vdash$ "$A^{p_1} \subseteq \tau''(\xi)$", and an open set $U_1 \supset A^{p_1}$ such that $p_1 \Vdash$ "$\tau''(\xi) \setminus (U_0 \cup U_1)$ is uncountable".

This describes the inductive steps of the construction, and finishes the proof of Claim 3. $\square$

Define $p_\omega := \bigcup_{n<\omega} p_n$. We have

$$p_\omega \Vdash \text{“}\xi \text{ is first countable and countably compact”},$$

so that

$$p_\omega \Vdash \{\eta_n\colon n \in \omega\}^d \neq \emptyset. \tag{$*$}$$

Claim 4. *There is a point $z \in X$ and a point $\eta \in \xi$ such that the sequence $\langle x_n\colon n < \omega\rangle$ accumulate x to z, and such that $p_\omega \Vdash$ "$\tau(\eta) = \check{z}$".*

Proof. It follows from $(*)$ and the maximality principle that there is a name η and a sequence of names $\{\eta_{n_k}\colon k \in \omega\} \subseteq \{\eta_n\colon n \in \omega\}$ such that

$$p_\omega \Vdash \eta \in \xi \text{ and } \eta_{n_k} \text{ converge to } \eta.$$

Since X is countably compact, there must be a subspace of the $\{x_n\colon n \in \omega\}$ say $\{x_{n_k}\colon k \in \omega\}$ that converges in X to a point z. Then z and η are as required. □

It follows, on the one hand, that $z \in A_{p_\omega}$, and on the other hand, $z \notin A^{p_n}$ for all $n \in \omega$. This is because $A^{p_n} \subset U_n$, U_n is open, and $x_m \notin U_n$ $\forall m > n$, while some infinite subsequence of $\{x_n\}$ converges to z. By Claim 4 and the definition of A^{p_n} we must have that $z \notin dom(p_n)$ for all n. Therefore, $z \notin dom\, p_\omega$. Now, we are in the situation where $z \in A_{p_\omega} \setminus dom\, p_\omega$. The sought after condition is $p_\omega \cup \{\langle z, 1\rangle\}$, and this proves the proposition. □

Corollary 3. *For every topological space X there is a forcing notion $\mathbb{P}$ which forces that $X \nrightarrow (Y)^1_2$ for any first countable uncountable countably compact space Y. Indeed, one has*: $X \nrightarrow (Y_1, Y_2)^1$ *for first countable uncountable countably compact spaces Y_1 and Y_2.*

Since our forcing notion is σ-closed, we know that certain spaces will remain the same in the extension. Examples of such spaces are Σ-products of any topological space. Hence we have the consistency of the following negative relations.

Corollary 4. *There is a forcing extension in which we have*
(1) $\Sigma_\kappa \nrightarrow (\omega_1)^1_2$.
(2) $\Sigma_\kappa \nrightarrow (2^\omega)^1_2$.
(3) $\Sigma_\kappa \nrightarrow (2^\omega, \omega_1)^1$.

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